Algebra, Number theory and Combinatorics Team Contest

Question 1. Fix positive integers $0 < k < n$. Denote by $V_{k,n}$ the space of $k \times n$ real matrices of rank k. Given $A \in V_{k,n}$, we write $A = \begin{pmatrix} v_1 & \cdots & v_i & \cdots & v_n \end{pmatrix}$ where v_i is the i-th column vector of A. Let

$$A_\infty = \begin{pmatrix} \cdots & v_1 & \cdots & v_i & \cdots & v_n & \cdots \end{pmatrix}$$

be the $k \times \infty$ matrix that is the cyclic extension of A, i.e. $v_{i+dn} = v_i$ for any $d \in \mathbb{Z}$. Define a function $f_A : \mathbb{Z} \to \mathbb{Z}$ by

$$f_A(i) = \min\{j \geq i | v_i \in \mathrm{Span}(v_{i+1}, v_{i+2}, \ldots, v_j)\}.$$

Note that span of the empty set is the zero space. Show that

1. f_A is bijective.
2. $$\sum_{i=1}^{n} f_A(i) = \binom{n+1}{2} + k \cdot n.$$

Question 2. For p a prime number, we write $\mathbb{Q}_p$ the field of p-adic numbers.

1. Show that the polynomials $f(X) = X^4 + 9 \in \mathbb{Q}_3[X]$ and $g(X) = X^8 + 9 \in \mathbb{Q}_3[X]$ are irreducible.
2. Let $K/\mathbb{Q}_3$ be a splitting field of $g(X)$. Determine $[K : \mathbb{Q}_3]$.

Question 3. Prove that $GL_n(\mathbb{Z})$ contains, up to isomorphism, only finitely many finite subgroups.